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THE EXTERIOR DERIVATIVE AS A KILLING VECTOR FIELD*

BY

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ABSTRACT

Among all the homogeneous Riemannian graded metrics on the algebra of differential forms, those for which the exterior derivative is a Killing graded vector field are characterized. It is shown that all of them are odd, and are naturally associated to an underlying smooth Riemannian metric. It is also shown that all of them are Ricci-flat in the graded sense, and have a graded Laplacian operator that annihilates the whole algebra of differential forms.

1. Introduction

Graded manifold theory, as developed for example in [4], provides a natural framework to address some geometrical questions that arose from the study of the de Rham complex of differential forms on a smooth manifold M. If M is an n-dimensional smooth manifold, and $\Omega(M)$ is its corresponding $\mathbb{Z}_2$-graded-commutative $\mathbb{R}$-algebra of differential forms, the pair $(M, \Omega(M))$ is an (n,n)-dimensional $\mathbb{Z}_2$-graded manifold (*graded manifold* for short). Abstractly, graded

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manifold theory treats the pair $(M, \Omega(M))$ as a ringed space; the ring $\Omega(M)$ is then the ring of $\mathbb{Z}_2$-graded (or super) functions on $(M, \Omega(M))$. This enlarges the ring $C^\infty(M)$ of smooth functions, as $\Omega(M) \supset \Omega^0(M) = C^\infty(M)$. Furthermore, by definition, the canonical projection of $\Omega(M)$ onto the ring of residues modulo the ideal of nilpotents gives a canonical embedding $(M, C^\infty(M)) \hookrightarrow (M, \Omega(M))$.

Now, vector fields on $(M, \Omega(M))$ are identified with the ($\mathbb{Z}_2$-) graded derivations of $\Omega(M)$. For example, the ordinary exterior derivative d is such a derivation. When d is regarded as a vector field, it makes sense to ask when is it a Killing vector field for a given $\mathbb{Z}_2$-graded metric on $(M, \Omega(M))$. A ($\mathbb{Z}_2$-) graded metric on $(M, \Omega(M))$ is an $\Omega(M)$-bilinear pairing,

$$\langle \cdot, \cdot \rangle \colon \operatorname{Der} \Omega(M) \times \operatorname{Der} \Omega(M) \to \Omega(M)$$

satisfying appropriate conditions (cf. §2 below). The purpose of this work is to determine all those graded metrics such that, with respect to the usual $\mathbb{Z}$-gradings on $\operatorname{Der} \Omega(M)$, and $\Omega(M)$, the pairing $\langle \cdot, \cdot \rangle$ is homogeneous of $\mathbb{Z}$-degree $+1$, and with respect to the $\mathbb{Z}_2$-gradings d is an infinitesimal superisometry for it; i.e.,

$$\langle [\mathrm{d}, D_1], D_2 \rangle + (-1)^{|D_1|} \langle D_1, [\mathrm{d}, D_2] \rangle = \mathrm{d} \langle D_1, D_2 \rangle$$

for all $\mathbb{Z}_2$-graded derivations D_1, and D_2 of $\Omega(M)$, $|D_1|$ being the $\mathbb{Z}_2$-degree of homogeneity of D_1.

Now, we have shown in Proposition 3.2 below that there are no even graded metrics having the exterior derivative as a Killing graded vector field. Nevertheless, we have found a wide class of graded metrics for which the conditions above are satisfied; namely, the class of odd graded metrics defined by a Riemannian metric on the base manifold M by means of a canonical construction (cf. Proposition 3.3 below). This constitutes then the supersymmetric counterpart of a structure previously studied by Koszul (see [5]). There, the question was posed as to what graded Poisson brackets can be defined on the graded algebra of differential forms. It has been shown (see also [2]) that odd graded Poisson brackets of $\mathbb{Z}$-degree $+1$ associated to classical Poisson brackets are completely characterized by the property that d is a Poisson derivation.

Having determined a class of metrics by such a ($\mathbb{Z}_2$-graded) geometrical property, we then compute some graded-Riemannian geometrical objects associated to the members of this class. In §4 and §5 below we show that all metrics of this kind are Ricci flat in the graded sense, and that their corresponding graded Laplacian operators vanish identically on the ring of super functions; i.e., any differential form becomes harmonic. We remark that these geometrical objects are computed with respect to the graded Levi-Civita connection. We have included a proof of its existence and uniqueness for a given graded metric (cf. 4.2 below). Our proof is intrinsic; it does not depend on local coordinates, nor on the fact that the graded metric is homogeneous. We also remark that the concept of graded connection we deal with is categorical for graded manifolds in general; it is different from the notion of superconnection introduced in [7]. The latter was meant as an odd derivation in $\Omega(M;TM)$—the $\Omega(M)$-module of differential forms with coefficients in the tangent bundle TM (see [7] for details).

2. Graded metrics

Let M be a smooth manifold of dimension n, and let $\Omega(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$ be its algebra of differential forms. This is a $\mathbb{Z}$-graded algebra, which becomes a $\mathbb{Z}_2$-graded algebra by considering the original grading mod 2. Graded manifold theory centers its attention in the latter, but we shall refer ourselves to both gradings. We shall adopt the convention that if v is an element of this or any other graded algebra or module, and the notation $|v|$ is used, we are tacitly assuming that v is homogeneous with respect to the $\mathbb{Z}_2$-grading. On the other hand, we shall occassionally need to refer ourselves to the $\mathbb{Z}$-degree of homogeneity of an element, in which case we shall explicitly emphasize the meaning of the notation $|v|$.

Let $\operatorname{Der}\Omega(M)$ be the left graded $\Omega(M)$-module of all derivations on $\Omega(M)$. $\operatorname{Der}\Omega(M)$ is a graded Lie algebra with the usual graded commutator (see [3] and [4]). It can also be regarded as a right graded $\Omega(M)$-module with multiplication $D\alpha = (-1)^{|\alpha||D|}\alpha D$. Actually, the assignment $U \mapsto \operatorname{Der}\Omega(U)$, for each open subset $U \subset M$, defines a locally free $\Omega(M)$-module of graded rank (n,n) with which the **graded vector fields** on the graded manifold $\big(M,\Omega(M)\big)$ are identified (cf. [4]).

Let $\operatorname{Hom}\big(\operatorname{Der}\Omega(M),\Omega(M)\big)$ be the right graded $\Omega(M)$-module of $\Omega(M)$-linear graded homomorphisms from the derivations $\operatorname{Der}\Omega(M)$ into the superfunctions $\Omega(M)$. This is the module of graded differential 1-forms on $\big(M,\Omega(M)\big)$. The action of a graded differential 1-form λ on a derivation D will be denoted by $\langle D;\lambda\rangle$, and for $\alpha \in \Omega(M)$, $\lambda\alpha$ is the homomorphism defined by $\langle D;\lambda\alpha\rangle = \langle D;\lambda\rangle\alpha$. It can also be regarded as a left graded $\Omega(M)$-module with multiplication $\alpha\lambda = (-1)^{|\alpha||\lambda|}\lambda\alpha$.

Definition 2.1: A graded metric on the algebra of differential forms is a graded symmetric, non-degenerate, bilinear map

$$G\colon \operatorname{Der}\Omega(M) \times \operatorname{Der}\Omega(M) \to \Omega(M),$$
$$(D_1, D_2) \to \langle D_1, D_2; G\rangle.$$

That is, a map satisfying the following conditions:

(1) $\langle D_1, D_2; G\rangle = (-1)^{|D_1||D_2|}\langle D_2, D_1; G\rangle$,

(2) $\langle \alpha D_1, D_2; G\rangle = \alpha\langle D_1, D_2; G\rangle = (-1)^{|D_1||\alpha|}\langle D_1, \alpha D_2; G\rangle$, $\alpha \in \Omega(M)$,

(3) The linear map $D \mapsto \langle D, \cdot\,; G\rangle$ is an isomorphism between the $\Omega(M)$-modules $\operatorname{Der}\Omega(M)$ and $\operatorname{Hom}(\operatorname{Der}\Omega(M), \Omega(M))$.

A graded metric is homogeneous of degree $k \in \mathbb{Z}$ if $|\langle D_1, D_2; G\rangle| = |D_1| + |D_2| + k$. A graded metric is even (resp. odd) if $|\langle D_1, D_2; G\rangle| = |D_1| + |D_2| \pmod 2$ (resp., $|\langle D_1, D_2; G\rangle| = |D_1| + |D_2| + 1 \pmod 2$).

Let U be an open coordinate neighborhood in M and let $\{X_1, \ldots, X_n\}$ be a local frame of vector fields in U. It is easy to check that $\{\mathcal{L}_{X_1}, \ldots, \mathcal{L}_{X_n}, i_{X_1}, \ldots, i_{X_n}\}$ is a local frame for $\operatorname{Der}\Omega(U)$ (cf. [3]). Thus, a graded metric is completely determined by its action on the pairs of derivations $(\mathcal{L}_X, \mathcal{L}_Y)$, $(\mathcal{L}_X, i_Y)$, and (i_X, i_Y) where X and Y are vector fields on M.

3. Graded metrics having the exterior derivative as a Killing graded vector field

Definition 3.1: A derivation $D \in \operatorname{Der}\Omega(M)$ is a **Killing graded vector field** for a graded metric G if

$$D\langle D_1, D_2; G\rangle = \langle [D, D_1], D_2; G\rangle + (-1)^{|D||D_1|}\langle D_1, [D, D_2]; G\rangle,$$

for all $D_1, D_2 \in \operatorname{Der}\Omega(M)$.

We shall now determine a class of matrics having the exterior derivative as a Killing graded vector field. We first turn our attention to *even* graded metrics:

PROPOSITION 3.2: *There are no even graded metrics having the exterior derivative as a Killing graded vector field.*

Proof: Let G be an even metric. Let $\pi_{(0)}\colon \Omega(M) \to \Omega^0(M) = C^\infty(M)$ be the projection map that assigns to each differential form, its component of $\mathbb{Z}$-degree

0. We may define a Riemannian metric g by $g(X,Y) = \pi_{(0)}(\langle \mathcal{L}_X, \mathcal{L}_Y; G\rangle)$, for any pair (X,Y) of vector fields on M.

Now, suppose the exterior derivative d is a Killing graded vector field for G. Then, 3.1 applied to the pair $(\mathcal{L}_X, \mathcal{L}_Y)$ says that $\mathrm{d}(\langle \mathcal{L}_X, \mathcal{L}_Y; G\rangle) = 0$. Therefore, $\mathrm{d}(g(X,Y)) = \mathrm{d}(\pi_{(0)}(\langle \mathcal{L}_X, \mathcal{L}_Y; G\rangle)) = 0$; in other words, $g(X,Y)$ is a constant function for any pair of vector fields (X,Y). Whence $g = 0$, in contradiction to the fact that g is a Riemannian metric. ∎

Thus, if such graded metrics actually exist in homogeneous form, they must be *odd*. The next result shows that there is at least a good supply of examples coming from ordinary Riemannian manifolds:

PROPOSITION 3.3: *There is a one-to-one correspondence between Riemannian metrics on M and graded metrics on $\Omega(M)$ of $\mathbb{Z}$-degree $+1$ having* d *as a Killing graded vector field. Specifically, given a Riemannian metric g on M its corresponding graded metric G is given by*

$$\begin{aligned}\langle \mathcal{L}_X, \mathcal{L}_Y; G\rangle &= \mathrm{d}(g(X,Y)),\\ \langle \mathcal{L}_X, i_Y; G\rangle &= \langle i_X, \mathcal{L}_Y; G\rangle = g(X,Y),\\ \langle i_X, i_Y; G\rangle &= 0.\end{aligned}$$

Proof: Let g be a Riemannian metric on M and let G be the odd graded metric defined as in the statement. An easy computation on pairs of derivations of the kind $\mathcal{L}_X$ and i_Y shows that the exterior derivative is a Killing graded vector field for G.

Conversely, let G be an odd metric of $\mathbb{Z}$-degree $+1$ for which d is Killing. Let X and Y be vector fields on M. Note first that the action of G on the pair of derivations of degree -1, (i_X, i_Y), must be a differential form of degree -1, so $\langle i_X, i_Y; G\rangle = 0$. Now, define $g \in \Gamma(T^*M \otimes T^*M)$ by $g(X,Y) = \langle \mathcal{L}_X, i_Y; G\rangle$. It follows, from (2) of 2.1, that g is tensorial. Furthermore,

$$\langle \mathcal{L}_X, i_Y; G\rangle - \langle \mathcal{L}_Y, i_X; G\rangle = \langle [\mathrm{d}, i_X], i_Y; G\rangle - \langle i_X, [\mathrm{d}, i_Y]; G\rangle = \mathrm{d}\,\langle i_X, i_Y; G\rangle = 0$$

where the hypothesis of d being Killing has been used. Whence, g is a symmetric tensor field. Note that the non-degeneracy of G implies that g is also non-degenerate; thus, g is a Riemannian metric. Then, 3.1 easily implies that $\mathrm{d}(g(X,Y)) = \langle \mathcal{L}_X, \mathcal{L}_Y; G\rangle$. ∎

Definition 3.4: Let g be a Riemannian metric on M. We define the odd graded metric, G_g, associated to g by the formulae given in the statement of the previous proposition.

Remark: As is well known from classical (i.e., non-graded) Riemannian Geometry, the integral flow of a Killing vector field is a one-parameter subgroup of isometries. This is also true in the graded case, but one must take good care of the meaning of *the integral curve* associated to the odd vector field d. We refer the reader to [6] for a discussion on integral flows of graded vector fields in general, and for the explicit expression of the integral flow of the odd field d in the graded manifold $(M, \Omega(M))$ (cf. example 3.7 therein).

4. Riemmanian elements associated to a graded metric

The first step towards the definition of the Riemannian elements associated to a graded metric is the concept of graded connection.

Definition 4.1: A graded connection on $\Omega(M)$ is a mapping,

$$\tilde{\nabla}\colon \operatorname{Der}\Omega(M) \times \operatorname{Der}\Omega(M) \to \operatorname{Der}\Omega(M),$$
$$(D_1, D_2) \mapsto (D_1, D_2; \tilde{\nabla})$$

satisfying the following conditions:

(1) $(D_1, D_2 + D_3; \tilde{\nabla}) = (D_1, D_2; \tilde{\nabla}) + (D_1, D_3; \tilde{\nabla})$,
(2) $(D_1 + D_2, D_3; \tilde{\nabla}) = (D_1, D_3; \tilde{\nabla}) + (D_2, D_3; \tilde{\nabla})$,
(3) $(\alpha D_1, D_2; \tilde{\nabla}) = \alpha(D_1, D_2; \tilde{\nabla})$,
(4) $(D_1, \alpha D_2; \tilde{\nabla}) = D_1(\alpha) D_2 + (-1)^{|D_1||\alpha|}\alpha(D_1, D_2; \tilde{\nabla})$.

A $\mathbb{Z}_2$-graded connection in $\Omega(M)$ is homogeneous of degree $|\tilde{\nabla}|$ if, for any pair (D_1, D_2) of homogeneous derivations, $(D_1, D_2; \tilde{\nabla})$ is homogeneous, and

$$|(D_1, D_2; \tilde{\nabla})| = |D_1| + |D_2| + |\tilde{\nabla}|, \qquad \text{in } \mathbb{Z}_2.$$

Remark: We have made a change with respect to the usual notation in order to avoid a proliferation of signs and other complications appearing because of the grading of the algebra, and the linearity properties on homogeneous elements. Such complications, however, do not appear when the connection is even and in that case we will switch back to the usual notation, $\tilde{\nabla}_{D_1} D_2 = (D_1, D_2; \tilde{\nabla})$.

The **torsion**, T, of a graded connection is defined by

$$\langle D_1, D_2; T\rangle = (D_1, D_2; \tilde{\nabla}) - (-1)^{|D_1||D_2|}(D_2, D_1; \tilde{\nabla}) - [D_1, D_2].$$

Let G be a graded metric on $\Omega(M)$, and $\widetilde{\nabla}$ a graded connection. Write $\widetilde{\nabla} = \widetilde{\nabla}^0 + \widetilde{\nabla}^1$ for the decomposition of the graded connection into its $\mathbb{Z}_2$-homogeneous components. Then, $\widetilde{\nabla}$ is **metric** if

$$\begin{aligned} D(\langle D_1, D_2; G\rangle) =& \langle (D, D_1; \widetilde{\nabla}), D_2; G\rangle \\ &+ (-1)^{|D_1||D|}\langle D_1, (D, D_2; \widetilde{\nabla}^0); G\rangle + (-1)^{|D_1|(|D|+1)}\langle D_1, (D, D_2; \widetilde{\nabla}^1); G\rangle \end{aligned}$$

for all homogeneous derivations D, D_1, and D_2.

As in the classical case, any graded metric has an associated Levi-Civita connection.

THEOREM 4.2: *Given a graded metric, there is a unique torsionless and metric graded connection.*

Proof: Such a connection is given by the formula,

$$\begin{aligned} 2\langle (D_1, D_2; \widetilde{\nabla}), D_3; G\rangle =& D_1\langle D_2, D_3; G\rangle - (-1)^{|D_3|(|D_1|+|D_2|)} D_3\langle D_1, D_2; G\rangle \\ &+ (-1)^{|D_1|(|D_2|+|D_3|)} D_2\langle D_3, D_1; G\rangle + \langle [D_1, D_2], D_3; G\rangle \\ &- (-1)^{|D_1|(|D_2|+|D_3|)}\langle [D_2, D_3], D_1; G\rangle + (-1)^{|D_3|(|D_1|+|D_2|)}\langle [D_3, D_1], D_2; G\rangle. \end{aligned}$$

Indeed, if $\widetilde{\nabla}$ is a graded metric connection, then

$$\begin{aligned} D_1\langle D_2, D_3\rangle =& \langle (D_1, D_2; \widetilde{\nabla}), D_3\rangle \\ &+ (-1)^{|D_1||D_2|}\langle D_2, (D_1, D_3; \widetilde{\nabla}^0)\rangle + (-1)^{(|D_1|+1)|D_2|}\langle D_2, (D_1, D_3; \widetilde{\nabla}^1)\rangle, \end{aligned}$$

where $\widetilde{\nabla}^0$ and $\widetilde{\nabla}^1$ are, respectively, the even and odd parts of $\widetilde{\nabla}$ as above, and we have omitted the explicit reference to the metric G in order to keep the notation simpler. Now, the "cyclic sum"

$$D_1\langle D_2, D_3\rangle + (-1)^{|D_1|(|D_2|+|D_3|)} D_2\langle D_3, D_1\rangle - (-1)^{|D_3|(|D_1|+|D_2|)} D_3\langle D_1, D_2\rangle$$

is equal to

$$\begin{aligned} &\langle (D_1, D_2; \widetilde{\nabla}) + (-1)^{|D_1||D_2|}((D_2, D_1; \widetilde{\nabla}^0) + (D_2, D_1; \widetilde{\nabla}^1)), D_3\rangle \\ &+ (-1)^{|D_2||D_3|}\langle ((D_1, D_3; \widetilde{\nabla}^0) + (D_1, D_3; \widetilde{\nabla}^1)) - (-1)^{|D_1||D_3|}(D_3, D_1; \widetilde{\nabla}), D_2\rangle \\ &+ (-1)^{|D_1|(|D_2|+|D_3|)}\langle (D_2, D_3; \widetilde{\nabla}) - (-1)^{|D_2||D_3|}((D_3, D_2; \widetilde{\nabla}^0) + (D_3, D_2; \widetilde{\nabla}^1)), D_1\rangle \end{aligned}$$

and using now the fact that $\widetilde{\nabla}$ is a torsionless connection, this simplifies to

$$\begin{aligned} 2\langle (D_1, D_2; \widetilde{\nabla}), D_3\rangle - \langle [D_1, D_2], D_3\rangle + (-1)^{|D_2||D_3|}\langle [D_1, D_3], D_2\rangle \\ + (-1)^{|D_1|(|D_2|+|D_3|)}\langle [D_2, D_3], D_1\rangle \end{aligned}$$

from which the formula for (and consequentlly, the uniqueness of) the graded linear connection is established. It is now a matter of simple computation to prove that this connection is metric and torsionless. ∎

Remark: The formula we have derived for the graded Levi-Civita connection coincides with that obtained in [1] for even metrics on an arbitrary supermanifold. Also, an expression in local coordinates can be found in [8]. Our proof does not depend on local coordinates, nor on the fact that the graded metric is homogeneous.

Remark: Note that the graded Levi-Civita connection for a homogeneous graded metric (i.e., odd or even) is always even. From now on we shall work exclusively with homogeneous graded metrics and we shall use the classical notation for the corresponding Levi-Civita connection; namely, $(D_1, D_2; \tilde{\nabla}) = \tilde{\nabla}_{D_1} D_2$.

Now, the **graded curvature tensor** of $\tilde{\nabla}$ is defined by

$$R^G(D_1, D_2)D_3 = [\tilde{\nabla}_{D_1}, \tilde{\nabla}_{D_2}]D_3 - \tilde{\nabla}_{[D_1, D_2]}D_3.$$

The **graded Ricci tensor** is the graded symmetric bilinear mapping defined by

$$S^G(D_1, D_2) = \mathrm{sTr}\,(D \mapsto R^G(D_1, D)D_2),$$

where sTr denotes the supertrace of the given endomorphism.

The supertrace of any endomorphism H of $\mathrm{Der}\,\Omega(M)$ can be computed with the aid of the odd graded metric G_g in the following manner. Let $\{X_k\}_{k=1}^n$ be an orthonormal frame for g. Then, $\{\mathcal{L}_{X_k}, i_{X_k}\}_{k=1}^n$ is a basis of graded derivations that satisfies the following relations,

$$\langle \mathcal{L}_{X_k}, \mathcal{L}_{X_\ell}; G_g\rangle = 0 = \langle i_{X_k}, i_{X_\ell}; G_g\rangle, \qquad \langle \mathcal{L}_{X_k}, i_{X_\ell}; G_g\rangle = \delta_{k\ell} = \langle i_{X_\ell}, \mathcal{L}_{X_k}; G_g\rangle,$$

and therefore

$$\mathrm{sTr}(H) = \sum_{k=1}^{n} \langle H(\mathcal{L}_{X_k}), i_{X_k}; G_g\rangle - \langle H(i_{X_k}), \mathcal{L}_{X_k}; G_g\rangle.$$

THEOREM 4.3: *The graded metric G_g is Ricci flat.*

Proof: The Levi-Civita connection associated to G_g is given by

$$\tilde{\nabla}_{\mathcal{L}_X} \nabla_Y = \mathcal{L}_{\nabla_X Y},\ \tilde{\nabla}_{\mathcal{L}_X} i_Y = i_{\nabla_X Y},\ \tilde{\nabla}_{i_X} \mathcal{L}_Y = i_{\nabla_X Y},\ \tilde{\nabla}_{i_X} i_Y = 0,$$

where ∇ and R^g are the Levi-Civita connection and the curvature tensor of g. Now the graded curvature tensor of G_g is given by

$$R(\mathcal{L}_X, \mathcal{L}_Y)\mathcal{L}_Z = \mathcal{L}_{R^g(X,Y)Z},$$
$$R(\mathcal{L}_X, \mathcal{L}_Y)i_Z = i_{R^g(X,Y)Z}, \quad R(\mathcal{L}_X, i_Y)\mathcal{L}_Z = i_{R^g(X,Y)Z},$$
$$R(\mathcal{L}_X, i_Y)i_Z = 0, \quad R(i_X, i_Y) = 0.$$

Using the above formula for the supertrace it is easy to verify that the Ricci tensor vanishes. ∎

5. Gradient, divergence, and Laplacian operators for G_g

Definition 5.1: Let G be a graded metric. Define the graded musical isomorphisms with respect to G by

$$\flat\colon \operatorname{Der}\Omega(M) \to \operatorname{Hom}(\operatorname{Der}\Omega(M), \Omega(M)),$$

$$D^\flat = \langle \quad , D; G\rangle,$$

and let

$$\sharp\colon \operatorname{Hom}(\operatorname{Der}\Omega(M), \Omega(M)) \to \operatorname{Der}\Omega(M)$$

be the inverse of $\flat$.

The proof of the following lemma is a straightforward routine.

Lemma 5.2: *For any $D \in \operatorname{Der}\Omega(M)$, $\lambda \in \operatorname{Hom}(\operatorname{Der}\Omega(M), \Omega(M))$ and $\alpha \in \Omega(M)$, we have*

(1) $|D^\flat| = |D| + |G|$ *and* $|\lambda^\sharp| = |\lambda| - |G|$.
(2) $(\alpha D)^\flat = (-1)^{|\alpha|(|D|+|G|)} D^\flat \alpha$.
(3) $(\lambda\alpha)^\sharp = (-1)^{|\alpha||\lambda|} \alpha\lambda^\sharp$.

Definition 5.3: Let G be a graded metric and let α be a differential form on M. The **graded gradient** of α is the unique graded vector field, $\operatorname{Grad}^G \alpha$, such that

$$\langle D, \operatorname{Grad}^G \alpha; G\rangle = D(\alpha),$$

for all $D \in \operatorname{Der}\Omega(M)$.

Again, the proof of the following lemma consists of a straightforward verification from the definitions.

LEMMA 5.4: *Let G be a graded metric and let D be a Killing vector field for it. Then, for any $\alpha \in \Omega(M)$,*

$$[D, \operatorname{Grad}^G(\alpha)] = \operatorname{Grad}^G(D\alpha).$$

A Riemannian metric g on M defines an isomorphism $g^\sharp\colon T^*M \to TM$. This isomorphism can be uniquely extended to a derivation of $\mathbb{Z}$-degree -1 (denoted by the same symbol), $g^\sharp\colon \Omega(M) \to \Omega(M, TM)$, satisfying $g^\sharp(f) = 0$ for all $f \in \Omega^0(M)$.

Now, a well known result of [3] provides a unique decomposition of any derivation into two terms: first, a derivation that commutes with the exterior derivative, and second, an algebraic derivation. The application of this result to the derivation $\operatorname{Grad}^G \alpha$ gives the following:

PROPOSITION 5.5: *Let g be a Riemannian metric on M, and let G_g be its associated odd graded metric. Then, for any $\alpha \in \Omega(M)$,*

$$\operatorname{Grad}^{G_g} \alpha = \mathcal{L}_{g^\sharp(\alpha)} + i_{g^\sharp(\mathrm{d}\alpha)}.$$

In particular, for any $f \in \Omega^0(M) \simeq C^\infty(M)$,

$$\operatorname{Grad}^{G_g} f = i_{\mathbf{grad}\, f}, \quad \operatorname{Grad}^{G_g} \mathrm{d}f = \mathcal{L}_{\mathbf{grad}\, f},$$

where $\operatorname{grad} f$ *is the gradient with respect to the Riemannian metric g on M.*

Proof: It is easy to check that $\overline{\operatorname{Grad}^{G_g}}\colon \Omega(M) \to \operatorname{Der}\Omega(M)$, defined by

$$\overline{\operatorname{Grad}^{G_g}}(\alpha) = (-1)^{|\alpha|} \operatorname{Grad}^{G_g}(\alpha),$$

is a derivation of degree -1. This is a consequence of the following formula:

$$\operatorname{Grad}(\alpha\beta) = (-1)^{|\beta|} \operatorname{Grad}^{G_g}(\alpha)\beta + \alpha \operatorname{Grad}^{G_g}(\beta),$$

where the right module structure of $\operatorname{Der}\Omega(M)$ is given by $D\beta = (-1)^{|D||\beta|}\beta D$.

Let us suppose that $\operatorname{Grad}^{G_g}(\alpha) = \mathcal{L}_{K_\alpha} + i_{L_\alpha}$, where $K_\alpha \in \Omega^{|\alpha|-1}(M; TM)$ and $L_\alpha \in \Omega^{|\alpha|}(M; TM)$, $|\alpha|$ being the $\mathbb{Z}$-degree of α. Then, by application of Lemma 5.4, and having in mind that d is a Killing vector field, it follows that for any $\alpha \in \Omega(M)$,

$$L_\alpha = K_{\mathrm{d}\alpha}.$$

Now, as a consequence of the fact that $\overline{\mathrm{Grad}^{G_g}}$ is a derivation of degree -1, we have that the operator $\bar{K}\colon \Omega(M) \to \Omega(M;TM)$ is also a derivation of degree -1. Thus, it is completely determined by its action on $\Omega^0(M) + \mathrm{d}\,\Omega^0(M)$. Let f be a smooth function. Then

$$X(f) = \langle i_X, \mathrm{Grad}^{G_g}(\mathrm{d}f)\rangle = g(X, K_{\mathrm{d}f}).$$

Therefore $K_{\mathrm{d}f} = \mathrm{grad}\, f$, and

$$\mathrm{Grad}^{G_g}(\mathrm{d}f) = \mathcal{L}_{\mathrm{grad}\, f}.$$

Moreover,

$$0 = \langle i_X, \mathrm{Grad}^{G_g}(f)\rangle = g(X, K_f).$$

Hence $K_f = 0$, and therefore $\mathrm{Grad}^{G_g}(f) = i_{\mathrm{grad}\, f}$. Finally, the derivation $\bar{K}$ is completely determined by $\bar{K}_f = 0$ and $\bar{K}_{\mathrm{d}f} = -g^\sharp(\mathrm{d}f)$, which proves the result. ∎

Let G be a graded metric. We shall now define the divergence operator acting on graded 1-forms. Let λ be a graded 1-form, then $\widetilde{\nabla}\lambda$ can be considered as a map from $\mathrm{Der}\,\Omega(M)$ into $\mathrm{Hom}(\mathrm{Der}\,\Omega(M), \Omega(M))$. Then $(\widetilde{\nabla}\lambda)^\sharp$ is a map of $\mathrm{Der}\,\Omega(M)$ into itself, but it is not $\Omega(M)$-linear. In order to get an $\Omega(M)$-linear morphism we have to introduce a sign. Let

$$H_\lambda\colon \mathrm{Der}\,\Omega(M) \to \mathrm{Der}\,\Omega(M)$$

be the endomorphism defined by

$$D \quad \mapsto \quad \langle D; H_\lambda\rangle = (-1)^{|D|(|\lambda|+|G|)}(\widetilde{\nabla}_D\lambda)^\sharp.$$

Definition 5.6: The **graded divergence** of λ is defined by

$$\delta^G\lambda = -\mathrm{sTr}(H_\lambda),$$

where sTr denotes the supertrace.

The graded divergence operator $\mathrm{Div}^G\colon \mathrm{Der}\,\Omega(M) \to \Omega(M)$ is then defined by

$$\mathrm{Div}^G(D) = -\delta^G(D^\flat).$$

LEMMA 5.7: *Let g be a Riemannian metric on M and let G_g be its associated odd metric. If $\{X_k\}_{k=1}^n$ is an orthonormal basis for g, then*

$$\delta^{G_g}\lambda = (-1)^{|\lambda|}\sum_{k=1}^{n}\mathcal{L}_{X_k}(\langle i_{X_k};\lambda\rangle) - i_{X_k}(\langle \mathcal{L}_{X_k};\lambda\rangle).$$

Proof: If $\{X_k\}_{k=1}^n$ is an orthonormal basis for g, then

$$\mathrm{sTr}\,(H_\lambda) = \sum_{k=1}^{n}\langle(\widetilde{\nabla}_{\mathcal{L}_{X_k}}\lambda)^\sharp, i_{X_k}; G_g\rangle - (-1)^{|\lambda|}\langle(\widetilde{\nabla}_{i_{X_k}}\lambda)^\sharp, \mathcal{L}_{X_k}; G_g\rangle.$$

Interchanging the arguments and applying the definition of the $\sharp$ morphism we obtain

$$\begin{aligned}\mathrm{sTr}\,(H_\lambda) &= (-1)^{|\lambda|}\sum_{k=1}^{n}\langle i_{X_k}, (\widetilde{\nabla}_{\mathcal{L}_{X_k}}\lambda)^\sharp; G_g\rangle - \langle \mathcal{L}_{X_k}, (\widetilde{\nabla}_{i_{X_k}}\lambda)^\sharp; G_g\rangle \\ &= (-1)^{|\lambda|}\sum_{k=1}^{n}\langle i_{X_k}; \widetilde{\nabla}_{\mathcal{L}_{X_k}}\lambda\rangle - \langle \mathcal{L}_{X_k}; \widetilde{\nabla}_{i_{X_k}}\lambda\rangle \\ &= (-1)^{|\lambda|}\sum_{k=1}^{n}\mathcal{L}_{X_k}\langle i_{X_k};\lambda\rangle - \langle\widetilde{\nabla}_{\mathcal{L}_{X_k}} i_{X_k};\lambda\rangle - i_{X_k}\langle\mathcal{L}_{X_k};\lambda\rangle + \langle\widetilde{\nabla}_{i_{X_k}}\mathcal{L}_{X_k};\lambda\rangle \\ &= (-1)^{|\lambda|}\sum_{k=1}^{n}\mathcal{L}_{X_k}\langle i_{X_k};\lambda\rangle - i_{X_k}\langle\mathcal{L}_{X_k};\lambda\rangle. \qquad \blacksquare\end{aligned}$$

PROPOSITION 5.8: *Let g be a Riemannian metric on M and let G_g be its associated odd metric. Then $\mathrm{Div}^{G_g}(i_X) = 0$ and $\mathrm{Div}^{G_g}(\mathcal{L}_X) = 0$, for any vector field X on M.*

Proof: This is a consequence of the previous lemma. ∎

For the graded manifold $(M, \Omega(M))$, the ring of "functions" is the algebra of differential forms on M. Then, the definition of the graded Laplacian on "functions" gives a classical (not graded) differential operator of order 2 on the algebra of differential forms. We now need to recall the definition of the **graded exterior derivative**: Given $\alpha \in \Omega(M)$, the graded exterior derivative of α, $\mathrm{d}^{gr}\alpha \in \mathrm{Hom}(\mathrm{Der}\,\Omega(M), \Omega(M))$, is defined by $\langle D; \mathrm{d}^{gr}\alpha\rangle = D(\alpha)$ for any $D \in \mathrm{Der}\,\Omega(M)$ (cf. [4]).

Definition 5.9: The **graded Laplacian operator**, Δ^G, for the graded metric G, is the differential operator defined by $\Delta^G\alpha = \delta^G(d^{gr}\alpha)$, for any $\alpha \in \Omega(M)$.

It is easy to check that $\Delta^G\alpha = -\,\mathrm{Div}^G(\mathrm{Grad}^G(\alpha))$.

THEOREM 5.10: *Let g be a Riemannian metric on M and let G_g be its associated odd metric. Then,* $\Delta^{G_g} = 0$.

Proof: Let α be a differential form and let $\{X_k\}_{k=1}^n$ be a g-orthonormal basis. Then, by Lemma 5.7,

$$\begin{aligned}\Delta^{G_g}(\alpha) &= \delta^G(d^{\text{gr}}\alpha)\\ &= (-1)^{|\alpha|}\sum_{k=1}^{n} \mathcal{L}_{X_k}(\langle i_{X_k}; d^{\text{gr}}\alpha\rangle) - i_{X_k}(\langle \mathcal{L}_{X_k}; d^{\text{gr}}\alpha\rangle)\\ &= (-1)^{|\alpha|}\sum_{k=1}^{n} \mathcal{L}_{X_k} i_{X_k}(\alpha) - i_{X_k}\mathcal{L}_{X_k}(\alpha)\\ &= (-1)^{|\alpha|}\sum_{k=1}^{n}[\mathcal{L}_{X_k}, i_{X_k}](\alpha) = 0. \qquad \blacksquare\end{aligned}$$

A consequence of this fact is that, at least for the odd case under consideration, finiteness theorems about the dimension of the spaces of harmonic forms are no longer true. For these odd metrics, any differential form is harmonic.

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